

Foundations of Derivative Review

Here is review of the foundations of differential calculus:

For a given function, $y = f(x)$, we want to study the slope at a point (which gives the rate at which the function is changing and is very helpful in many applications as we will see throughout the term). But we can't define slope using only one point; we need two points (x_1, y_1) and (x_2, y_2) in order to get slope = $\frac{y_2 - y_1}{x_2 - x_1}$.

The strategy we employed was to let $x_1 = x$ to $x_2 = x + h$.

(x is the place where we want the slope and h is a small number so that $x + h$ is a value close by).

Then we compute $y_1 = f(x)$ and $y_2 = f(x + h)$. And we get

$$\text{'slope of the secant line from } x \text{ to } x + h\text{' } = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}.$$

We then defined

$$\text{'slope of the tangent line at } x\text{' } = f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

We call this new function the derivative function. And, in chapter 3, we are making use of the definition to find shortcut rules for many of our functions. Here are some examples of how we used this definition and the observations we made in order to find short-cuts:

1. If $f(x) = x^n$, then $f'(x) = \lim_{h \rightarrow 0} \frac{(x + h)^n - (x)^n}{h}$.

Using the binomial theorem, we showed that this expression can be expanded and always gives $f'(x) = nx^{n-1}$.

2. If $f(x) = a^x$, then $f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$.

By experimentation, we found that this limit always approached $a^x \ln(a)$. Thus, we learned that $f'(x) = a^x \ln(a)$ and we made particular note that if $f(x) = e^x$, then $f'(x) = e^x$ (because $\ln(e) = 1$).

3. If $f(x) = \sin(x)$, then $f'(x) = \lim_{h \rightarrow 0} \frac{\sin(x + h) - \sin(x)}{h}$.

Using the sum identity for $\sin(x + h)$ we got: $\lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}$.

Then rearranging we got: $\lim_{h \rightarrow 0} \left(\cos(x) \frac{\sin(h)}{h} + \sin(x) \frac{\cos(h) - 1}{h} \right)$.

Then we explained how $\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$ and $\lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} = 0$.

So we got $f'(x) = \cos(x)$.

4. If $f(x) = \cos(x)$, then by a similar argument, you get $f'(x) = -\sin(x)$.

We will explore other functions and rules. But hopefully this reminds you of what derivatives are and where the shortcut rules come from.